SIXTH TERM EXAMINATION PAPERS

administered by the Oxford and Cambridge Schools Examination Board on behalf of the Cambridge Colleges

9475

MATHEMATICS III

Friday 28 June 1996, afternoon

3 hours

Additional materials:
script paper; graph paper; MF(STEP)1.
To be brought by candidate: electronic calculator; standard geometrical instruments.

All questions carry equal weight.

You are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

Candidates may attempt as many questions as they wish but marks will be assessed on the **six** *questions best answered.*

You are provided with Mathematical Formulae and Tables MF(STEP)1.

Electronic calculators may be used.

Section A: Pure Mathematics

1 Define $\cosh x$ and $\sinh x$ in terms of exponentials and prove, from your definitions, that

$$\cosh^4 x - \sinh^4 x = \cosh 2x$$

and

$$\cosh^4 x + \sinh^4 x = \tfrac{1}{4}\cosh 4x + \tfrac{3}{4}.$$

Find $a_0, a_1, \ldots, a_n$ in terms of n such that

$$\cosh^n x = a_0 + a_1 \cosh x + a_2 \cosh 2x + \cdots + a_n \cosh nx.$$

Hence, or otherwise, find expressions for $\cosh^{2m} x - \sinh^{2m} x$ and $\cosh^{2m} x + \sinh^{2m} x$, in terms of $\cosh kx$, where $k = 0, \cdots, 2m$.

2 For all values of a and b, either solve the simultaneous equations

$$\begin{aligned} x + y + az &= 2 \\ x + ay + z &= 2 \\ 2x + y + z &= 2b \end{aligned}$$

or prove that they have no solution.

3 Find

$$\int_0^\theta \frac{1}{1 - a\cos x}\,\mathrm{d}x,$$

where $0 < \theta < \pi$ and $-1 < a < 1$.

Hence show that

$$\int_0^{\pi/2} \frac{1}{2 - a\cos x}\,\mathrm{d}x = \frac{2}{\sqrt{4 - a^2}}\tan^{-1}\sqrt{\frac{2 + a}{2 - a}}\,,$$

and also that

$$\int_0^{3\pi/4} \frac{1}{\sqrt{2} + \cos x}\,\mathrm{d}x = \frac{\pi}{2}.$$

4 Find the integers k satisfying the inequality $k \leqslant 2(k-2)$.

Given that N is a strictly positive integer consider the problem of finding strictly positive integers whose sum is N and whose product is as large as possible. Call this largest possible product $P(N)$. Show that $P(5) = 2 \times 3$, $P(6) = 3^2$, $P(7) = 2^2 \times 3$, $P(8) = 2 \times 3^2$ and $P(9) = 3^3$.

Find $P(1000)$ explaining your reasoning carefully.

5 Show, using de Moivre's Theorem, or otherwise, that

$$\tan 7\theta = \frac{t(t^6 - 21t^4 + 35t^2 - 7)}{7t^6 - 35t^4 + 21t^2 - 1},$$

where $t = \tan\theta$.

(i) By considering the equation $\tan 7\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{7}\right),\ \tan^2\left(\frac{2\pi}{7}\right) \text{ and } \tan^2\left(\frac{3\pi}{7}\right)$$

and deduce the value of

$$\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right).$$

(ii) Find, without using a calculator, the value of

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right).$$

6 **(i)** Let S be the set of matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where a is any real non-zero number. Show that S is closed under matrix multiplication and, further, that S is a group under matrix multiplication.

(ii) Let G be a set of $n \times n$ matrices which is a group under matrix multiplication, with identity element $\mathbf{E}$. By considering equations of the form $\mathbf{BC} = \mathbf{D}$ for suitable elements $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of G, show that if a given element $\mathbf{A}$ of G is a singular matrix (i.e. $\det \mathbf{A} = 0$), then all elements of G are singular. Give, with justification, an example of such a group of singular matrices in the case $n = 3$.

7 **(i)** If $x + y + z = \alpha$, $xy + yz + zx = \beta$ and $xyz = \gamma$, find numbers A, B and C such that

$$x^3 + y^3 + z^3 = A\alpha^3 + B\alpha\beta + C\gamma.$$

Solve the equations

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 4. \end{aligned}$$

(ii) The area of a triangle whose sides are a, b and c is given by the formula

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter $\frac{1}{2}(a + b + c)$. If a, b and c are the roots of the equation

$$x^3 - 16x^2 + 81x - 128 = 0,$$

find the area of the triangle.

8 A transformation T of the real numbers is defined by

$$y = T(x) = \frac{ax - b}{cx - d},$$

where a, b, c, d are real numbers such that $ad \neq bc$. Find all numbers x such that $T(x) = x$. Show that the inverse operation, $x = T^{-1}(y)$ expressing x in terms of y is of the same form as T and find corresponding numbers a', b', c', d'.

Let S_r denote the set of all real numbers excluding r. Show that, if $c \neq 0$, there is a value of r such that T is defined for all $x \in S_r$ and find the image $T(S_r)$. What is the corresponding result if $c = 0$?

If T_1, given by numbers a_1, b_1, c_1, d_1, and T_2, given by numbers a_2, b_2, c_2, d_2, are two such transformations, show that their composition T_3, defined by $T_3(x) = T_2(T_1(x))$, is of the same form.

Find necessary and sufficient conditions on the numbers a, b, c, d for T^2, the composition of T with itself, to be the identity. Hence, or otherwise, find transformations T_1, T_2 and their composition T_3 such that ${T_1}^2$ and ${T_2}^2$ are each the identity but ${T_3}^2$ is not.

Section B: Mechanics

9 A particle of mass m is at rest on top of a smooth fixed sphere of radius a. Show that, if the particle is given a small displacement, it reaches the horizontal plane through the centre of the sphere at a distance

$$a(5\sqrt{5}+4\sqrt{23})/27$$

from the centre of the sphere.

[Air resistance should be neglected.]

10 Two rough solid circular cylinders, of equal radius and length and of uniform density, lie side by side on a rough plane inclined at an angle α to the horizontal, where $0<\alpha<\pi/2$. Their axes are horizontal and they touch along their entire length. The weight of the upper cylinder is W_1 and the coefficient of friction between it and the plane is μ_1. The corresponding quantities for the lower cylinder are W_2 and μ_2 respectively and the coefficient of friction between the two cylinders is μ. Show that for equilibrium to be possible:

(i) $W_1 > W_2$;

(ii) $\mu \geqslant \dfrac{W_1+W_2}{W_1-W_2}$;

(iii) $\mu_1 \geqslant \left(\dfrac{2W_1\cot\alpha}{W_1+W_2}-1\right)^{-1}$.

Find the similar inequality to (iii) for μ_2.

11 A smooth circular wire of radius a is held fixed in a vertical plane with light elastic strings of natural length a and modulus λ attached to the upper and lower extremities, A and C respectively, of the vertical diameter. The other ends of the two strings are attached to a small ring B of mass m which is free to slide on the wire. Show that, while both strings remain taut, the equation for the motion of the ring is

$$2ma\ddot{\theta}=\lambda(\cos\theta-\sin\theta)-mg\sin 2\theta,$$

where θ is the angle $\widehat{CAB}$.

Initially the system is at rest in equilibrium with $\sin\theta=3/5$. Deduce that $5\lambda=24mg$.

The ring is now displaced slightly. Show that in the ensuing motion it will oscillate with period approximately

$$10\pi\sqrt{\frac{a}{91g}}.$$

Section C: Probability & Statistics

12 It has been observed that Professor Ecks proves three types of theorems: 1, those that are correct and new; 2, those that are correct, but already known; 3, those that are false. It has also been observed that, if a certain of her theorems is of type i, then her next theorem is of type j with probability p_{ij}, where p_{ij} is the entry in the ith row and jth column of the following array:

$$\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

Let a_i, $i = 1, 2, 3$, be the probability that a given theorem is of type i, and let b_j be the consequent probability that the next theorem is of type j.

(i) Explain why $b_j = a_1 p_{1j} + a_2 p_{2j} + a_3 p_{3j}$.

(ii) Find values of a_1, a_2 and a_3 such that $b_i = a_i$ for $i = 1, 2, 3$.

(iii) For these values of the a_i find the probabilities q_{ij} that, if a particular theorem is of type j, then the *preceding* theorem was of type i.

13 Let X be a random variable which takes only the finite number of different possible real values x_1, x_2, $\ldots$, x_n. Define the expectation $\mathrm{E}(X)$ and the variance $\mathrm{var}(X)$ of X. Show that, if a and b are real numbers, then $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and express $\mathrm{var}(aX + b)$ similarly in terms of $\mathrm{var}(X)$.

Let λ be a positive real number. By considering the contribution to $\mathrm{var}(X)$ of those x_i for which $|x_i - \mathrm{E}(X)| \geqslant \lambda$, or otherwise, show that

$$\mathrm{P}[\,|X - \mathrm{E}(X)| \geqslant \lambda\,] \leqslant \frac{\mathrm{var}(X)}{\lambda^2}.$$

Let k be a real number satisfying $k \geqslant \lambda$. If $|\, x_i - \mathrm{E}(X)\,| \leqslant k$ for all i, show that

$$\mathrm{P}[\,|X - \mathrm{E}(X)| \geqslant \lambda\,] \geqslant \frac{\mathrm{var}(X) - \lambda^2}{k^2 - \lambda^2}.$$

14 Whenever I go cycling I start with my bike in good working order. However if all is well at time t, the probability that I get a puncture in the small interval $(t,\, t+\delta t)$ is $\alpha\,\delta t$. How many punctures can I expect to get on a journey during which my total cycling time is T?

When I get a puncture I stop immediately to repair it and the probability that, if I am repairing it at time t, the repair will be completed in time $(t,\, t+\delta t)$ is $\beta\,\delta t$. If $p(t)$ is the probability that I am repairing a puncture at time t, write down an equation relating $p(t)$ to $p(t+\delta t)$, and derive from this a differential equation relating $p'(t)$ and $p(t)$. Show that

$$p(t) = \frac{\alpha}{\alpha+\beta}(1-\exp(-(\alpha+\beta)t))$$

satisfies this differential equation with the appropriate initial condition.

Find an expression, involving α, β and T, for the time expected to be spent mending punctures during a journey of total time T. Hence, or otherwise, show that, the fraction of the journey expected to be spent mending punctures is given approximately by

$$\frac{\alpha T}{2} \quad \text{if } \ (\alpha+\beta)T \ \text{ is small,}$$

and by

$$\frac{\alpha}{\alpha+\beta} \quad \text{if } \ (\alpha+\beta)T \ \text{ is large.}$$